

# ON A THEOREM OF D. RYABOGIN AND V. YASKIN ABOUT DETECTING SYMMETRY

E. MAKAI, JR.\*, H. MARTINI, T. ÓDOR

ABSTRACT. We give a simple deduction of a recent theorem of D. Ryabogin and V. Yaskin, about detecting symmetry of star bodies in  $\mathbb{R}^n$  with  $C^1$  radial functions — via their conical section functions — from an older theorem of us.

## §1 NOTIONS AND NOTATIONS

We will work in *Euclidean space*  $\mathbb{R}^n$ , where  $n \geq 2$ . Its *unit sphere* will be written as  $S^{n-1}$ . We say that  $K \subset \mathbb{R}^n$  is a *star body* if it is of the form  $K = \{\lambda u \mid u \in S^{d-1}, 0 \leq \lambda \leq \varrho_K(u)\}$ , where  $\varrho_K : S^{n-1} \rightarrow (0, \infty)$  is a continuous function, which is called the *radial function* of the star body  $K$ . A *convex body in*  $\mathbb{R}^n$  is a compact convex set with interior points. If  $K \subset \mathbb{R}^n$  is a convex body containing 0 in its interior, then it is a star body. Moreover, its radial function  $\varrho_K$  is Lipschitz (we consider  $S^{n-1}$  with its geodesic metric, and Lipschitz is meant with respect to it), cf. [MMÓ1], first paragraph of §3. For  $f : S^{n-1} \rightarrow \mathbb{R}$  a Lipschitz function, we denote by  $L(f)$  its Lipschitz constant (with respect to the geodesic metric on  $S^{n-1}$ ). Observe that if the radial function  $\varrho_K$  of a star body  $K$  is  $C^1$  (for which we will shortly say that *the star body is  $C^1$* ), then it is Lipschitz (actually, this implication holds even for any function  $S^{n-1} \rightarrow \mathbb{R}$ ).

For  $\xi \in S^{n-1}$  we write  $\xi^\perp$  for the linear  $(n-1)$ -subspace of  $\mathbb{R}^n$  orthogonal to  $\xi$ . We will use also spherical polar coordinates, with north pole some  $\xi \in S^{n-1}$ . That is, we write each  $x \in S^{n-1}$  as

$$x = \xi \sin \psi + \eta \cos \psi, \quad \text{where } \eta \in S^{n-1} \cap \xi^\perp \quad \text{and} \quad -\pi/2 \leq \psi \leq \pi/2.$$

We call  $\psi$  the *geographic latitude* (which will be more convenient for us than the customarily used  $\varphi = \pi/2 - \psi$ ), and will write

$$x = (\eta, \psi).$$

A function  $f : S^{n-1} \rightarrow \mathbb{R}$  is *even* if  $f(x) = f(-x)$ , for all  $x \in S^{n-1}$ .

---

1991 *Mathematics Subject Classification*. 2010 *Mathematics Subject Classification*. Primary: 44A99. Secondary: 52A38.

*Key words and phrases*. conical section function, convex body, detecting evenness, detecting symmetry, linear integro-differential transformation, Lipschitz functions, radial function, star body.

\*Research (partially) supported by Hungarian National Foundation for Scientific Research, grant nos. K75016, K81146.

D. Ryabogin and V. Yaskin [RY], p. 509, denoted, for  $\xi \in S^{n-1}$  and  $z \in (-1, 1)$ , by  $C(\xi, z)$  the cone  $\{0\} \cup \{x \in \mathbb{R}^n \setminus \{0\} \mid \cos(\angle \xi 0x) = z\}$ . Then, for  $K \subset \mathbb{R}^n$  a star body, [RY], pp. 509-510, defined the *conical section function*  $C_{K,\xi}(z)$  of  $K$  as

$$C_{K,\xi}(z) := \text{vol}_{n-1}(K \cap C(\xi, z)),$$

where  $\text{vol}_{n-1}$  means  $(n-1)$ -volume.

## §2 SOME RESULTS OF D. RYABOGIN-V. YASKIN AND E. MAKAI, JR.-H. MARTINI-T. ÓDOR

[RY] proved the following geometrical theorem, by a relatively short proof, but using advanced methods, namely, Fourier transform techniques. (The converse implication in Theorem A is obvious.)

**Theorem A.** ([RY], Theorem 1.1) *Let  $K \subset \mathbb{R}^n$  be a  $C^1$  star body. Assume that, for all  $\xi \in S^{n-1}$ , the function  $C_{K,\xi}(z)$  has a critical point at  $z = 0$ . Then the body  $K$  is 0-symmetric.*

Here, a *critical point* of a function is a point such that the derivative of the function at this point exists, and equals 0.

[RY], in the remarks in the second paragraph after their Theorem 1.2, mentioned that, by the methods of [MMÓ1], Theorem A can be extended to any convex body containing 0 in its interior. This also follows from our Theorem A' below.

Theorem A follows from the following analytical theorem.

**Theorem A'.** *Let  $f : S^{n-1} \rightarrow \mathbb{R}$  be a Lipschitz function. Assume that, for almost all  $\xi \in S^{n-1}$ , we have that the integral of  $f$  on the set  $S^{n-1} \cap C(\xi, z)$ , as a function of  $z$ , has a critical point at  $z = 0$ . Then  $f$  is an even function.*

We obtain Theorem A, by applying Theorem A' to the  $C^1$ , hence Lipschitz function  $f := \varrho_K^{n-1}/(n-1)$ . Since the radial function  $\varrho_K$  of a convex body  $K \subset \mathbb{R}^d$ , containing 0 in its interior, is Lipschitz (cf. §1), the above mentioned extension of Theorem A to convex bodies, containing 0 in their interiors, follows from Theorem A' similarly.

**Remark.** To justify the hypotheses of Theorem A', we recall from [MMÓ1], Lemma 3.5 and its proof, and Lemma 3.6, the following. For  $f : S^{n-1} \rightarrow \mathbb{R}$  being a Lipschitz function, for almost all  $\xi \in S^{n-1}$  we have that, for almost all  $x \in S^{n-1} \cap \xi^\perp$ , the function  $f$  is differentiable. Further, for almost all  $\xi \in S^{n-1}$  we have that, for  $z = 0$ ,

$$\frac{d}{dz} \int_{S^{n-1} \cap (\xi^\perp + z\xi)} f(x) dx$$

exists. Moreover, it equals

$$\int_{S^{n-1} \cap \xi^\perp} \frac{\partial f}{\partial \psi}(x) dx,$$

where  $\psi = \psi_\xi$  is the geographic latitude, with the north pole at  $\xi$  (hence the partial derivative  $\partial f / \partial \psi$  is taken along a meridian, in the direction toward the north pole  $\xi$ ), and where also the second integral exists, for almost all  $\xi \in S^{n-1}$ . (These readily imply that, also in Theorem A', the converse implication holds.)

Now we cite a theorem from [MMÓ1].

**Theorem B.** ([MMÓ1], Lemma 3.6, Theorem 3.8) *Let  $f : S^{n-1} \rightarrow \mathbb{R}$  be a Lipschitz function. Assume that, for almost all  $\xi \in S^{n-1}$ , we have*

$$\int_{S^{n-1} \cap \xi^\perp} \frac{\partial f}{\partial \psi}(x) dx = 0.$$

*Then  $f$  is an even function.*

To justify the hypotheses of Theorem B, recall the Remark above. (The above remark readily implies that, also in Theorem B, the converse implication holds.)

We have used Theorem B in [MMÓ1] to prove another geometrical theorem. This theorem was proved for  $n = 2$  by [H], Theorem 1; for  $n \geq 3$  it was first proved by [MMÓ1], Corollary 3.4, Lemma 3.5, Theorem 3.8, by using spherical harmonics, and the Funk-Hecke formula. It was reproved, for  $n \geq 3$ , by a relatively short proof, however, using advanced methods, namely, Fourier transform techniques, by [RY], Theorem 1.2, for the  $C^1$  case. This geometrical theorem states the following. Let  $K \subset \mathbb{R}^n$  be a star body with Lipschitz radial function. Then, for almost all  $\xi \in S^{n-1}$ , the function  $z \mapsto \text{vol}_{n-1}(K \cap (\xi^\perp + z\xi))$  ( $\text{vol}_{n-1}$  meant here as  $(n-1)$ -dimensional Lebesgue measure) is differentiable at 0. Let, for almost all  $\xi \in S^{n-1}$ , this function have a critical point at  $z = 0$ . Then the body  $K$  is 0-symmetric. (The last but two sentence readily implies that, also in this geometrical theorem, the converse implication holds.)

An infinitesimal variant of the last mentioned geometrical theorem, for the case of a convex body (infinitesimally) close to the unit ball, not for the  $(n-1)$ -volumes of the intersections  $K \cap (\xi^\perp + z\xi)$ , but for the  $((n-2)$ -dimensional) surface area, and also for the lower (but positive) dimensional quermassintegrals (cf. [BF], §32, [Sch], §§ 4.1, 4.2) of these intersections, has been proved, in the sufficiently regular case, in [MM], Theorem. Details cf. there.

In [MMÓ2] we have proved an (almost) generalization of Theorem B, when  $\partial f / \partial \psi$  in the hypothesis of Theorem B was replaced by  $(\partial / \partial \psi)^m f$ , for  $m \geq 2$  an integer. Details cf. there.

In what follows, we show that our Theorem B implies Theorem A' (and thus also Theorem A).

### §3 PROOF OF THE IMPLICATION THEOREM B $\implies$ THEOREM A'.

*Proof.* We have, writing  $\sin \psi := z$  (where  $-\pi/2 < \psi < \pi/2$ ),

$$(1) \quad \int_{S^{n-1} \cap C(\xi, z)} f(x) dx = \int_{S^{n-1} \cap C(\xi, z)} f(\eta, \psi) d(\eta, \psi).$$

By the Lipschitz property of  $f$  we have, for  $(\eta, \psi) \in S^{n-1}$ ,

$$(2) \quad |f(\eta, \psi) - f(\eta, 0)| \leq L(f) \cdot |\psi|.$$

By (2) we have, for  $\psi \neq 0$ , that

$$(3) \quad \begin{cases} |f(\eta, \psi) - f(\eta, 0)|/|\sin \psi| = [|f(\eta, \psi) - f(\eta, 0)|/|\psi|] \cdot [\psi/\sin \psi] \\ < L(f) \cdot (\pi/2)/1 =: c. \end{cases}$$

By [MMÓ1], Lemma 3.5 and its proof, and Lemma 3.6 (cf. also our Remark), for almost all  $\xi \in S^{n-1}$ , we have that for almost all  $x \in S^{n-1} \cap \xi^\perp$  the function  $f$  is differentiable, and thus, in particular,  $(\partial f/\partial \psi)(x)$  (taken along a meridian, in the direction toward  $\xi$ ) exists. Moreover, for these (almost all)  $\xi$ 's, and for  $z \in (-1, 1) \setminus \{0\}$  and  $z \rightarrow 0$  we have, using in the first equality (1),

$$(4) \quad \begin{cases} \left[ \int_{S^{n-1} \cap C(\xi, z)} f(x) dx - \int_{S^{n-1} \cap C(\xi, 0)} f(x) dx \right] / z = \\ \left[ \int_{S^{n-1} \cap C(\xi, z)} f(\eta, \psi) d(\eta, \psi) - \int_{S^{n-1} \cap C(\xi, 0)} f(\eta, 0) d(\eta, 0) \right] / z = \\ \left[ \int_{S^{n-1} \cap \xi^\perp} f(\eta, \psi) d\eta \cdot \cos^{n-2} \psi - \int_{S^{n-1} \cap \xi^\perp} f(\eta, 0) d\eta \right] / \sin \psi = \\ \int_{S^{n-1} \cap \xi^\perp} [f(\eta, \psi) d\eta - f(\eta, 0)] / \psi \cdot [\psi / \sin \psi] d\eta + \\ \int_{S^{n-1} \cap \xi^\perp} f(\eta, \psi) d\eta \cdot (\cos^{n-2} \psi - 1) / \sin \psi \rightarrow \\ \int_{S^{n-1} \cap \xi^\perp} (\partial f / \partial \psi)(\eta) \cdot 1 \cdot d\eta + 0, \end{cases}$$

by  $\psi / \sin \psi \rightarrow 1$ , and  $|\int_{S^{n-1} \cap \xi^\perp} f(\eta, \psi) d\eta| \cdot (1 - \cos^{n-2} \psi) / |\sin \psi| \leq \cos^{n-2}(\psi) \times \text{vol}_{n-2}(S^{n-2}) \cdot \max\{|f(x)| \mid x \in S^{n-1}\} \cdot (1 - \cos^{n-2} \psi) / |\sin \psi| = O(|\psi|) \rightarrow 0$ , for  $\psi \rightarrow 0$  ( $\text{vol}_{n-2}$  meaning  $(n-2)$ -volume). Still we used for the convergence of the summand in the fourth line of (4) Lebesgue's dominated convergence theorem with integrable majorant  $c$ , cf. (3), for each  $\xi \in S^{n-1}$  for which for almost all  $x \in S^{n-1} \cap \xi^\perp$  the function  $f$  is differentiable, thus for almost all  $\xi \in S^{n-1}$ .

By the hypothesis of Theorem A', the last expression in (4) vanishes for almost all  $\xi \in S^{n-1}$ , thus the hypothesis of Theorem B is satisfied. Hence also the conclusion of Theorem B is satisfied, i.e.,  $f$  is even, which is the conclusion of Theorem A' as well. ■

## REFERENCES

- [BF] T. Bonnesen, W. Fenchel, *Theorie der konvexen Körper. Berichtigter Reprint*, Springer, Berlin-New York, 1974. MR **49**#9736.
- [H] P. C. Hammer, *Diameters of convex bodies*, Proc. Amer. Math. Soc. **5** (1954), 304-306. MR **15**,819.
- [MM] E. Makai, Jr., H. Martini, *Centrally symmetric convex bodies and sections having maximal quermassintegrals*, Studia Sci. Math. Hungar. **49** (2012), 189-199. MR **3058386**,
- [MMÓ1] E. Makai, Jr., H. Martini, T. Ódor, *Maximal sections and centrally symmetric bodies*, Mathematika **47** (2000), 19-30. MR **2003e**:52005.
- [MMÓ2] E. Makai, Jr., H. Martini, T. Ódor, *On an integro-differential transform on the sphere*, Studia Sci. Math. Hungar. **38** (2001), 299-312. MR **2003m**:44002.
- [RY] D. Ryabogin, V. Yaskin, *Detecting symmetry in star bodies*, J. Math. Anal. Appl. **395** (2012), 509-514. MR **2948241**.
- [Sch] R. Schneider, *Convex Bodies: the Brunn-Minkowski Theory*, Encyclopedia of Math. and its Appls., **44**, Cambridge Univ. Press, Cambridge, 1993. MR **94d**:52007.

ALFRÉD RÉNYI MATHEMATICAL INSTITUTE, HUNGARIAN ACADEMY OF SCIENCES,  
H-1364 BUDAPEST, PF. 127, HUNGARY <http://www.renyi.mta.hu/~makai>

FAKULTÄT FÜR MATHEMATIK, TECHNISCHE UNIVERSITÄT CHEMNITZ  
D-09107 CHEMNITZ, GERMANY

BOLYAI INSTITUTE, UNIVERSITY OF SZEGED,  
H-6720 SZEGED, ARADI VÉRTANÚK TERE 1, HUNGARY

*E-mail address:* [makai.endre@renyi.mta.hu](mailto:makai.endre@renyi.mta.hu),  
[martini@mathematik.tu-chemnitz.de](mailto:martini@mathematik.tu-chemnitz.de)  
[odor@math.u-szeged.hu](mailto:odor@math.u-szeged.hu)